FUNCTIONAL IMPLICIT OPERATIONS

BY

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ABSTRACT

Two forms of Keisler's characterization of functorial predicates are established for implicitly definable infinitary operations. In particular, functorial and implicitly definable ⇒ explicitly definable.

Let $\mathscr{S}^T$ be a variety of finitary algebras, σ an operation of T, Σ a set of operations of T such that every mapping of T-algebras which preserves all the operations in Σ preserves σ. Then T implies a theorem

$$x_0 = \sigma(x_1, \cdots, x_n) \Leftrightarrow (\exists t_1) \cdots (\exists t_k) \Phi,$$

where Φ is a conjunction of equations over Σ. Conversely, the displayed theorem implies the preservation relation. This is an easy and obvious conclusion from known results of E. W. Beth and H. J. Keisler [2].

Another easy result is that (in this setting) the displayed theorem is equivalent to a theorem of the form

$$x_0 = \sigma(x_1, \cdots, x_n) \Leftrightarrow (\forall t_0) \cdots (\forall t_k) [\Psi \Rightarrow t_0 = x_0],$$

Ψ like Φ. The purpose of this paper is to extend the results to infinitary algebras. The proofs are easier. The Beth and Keisler theorems are not consequences, because those theorems concern arbitrary finitary theories and predicates, not just equational theories and operations.

Though implicitly defined, σ thus turns out to be both existentially and universally definable. It need not have a quantifier-free definition, nor (in the finitary case) a definition in a language with a preassigned finite number of parameter

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symbols t_j, even if $n = 1$. Also, the universal form cannot be much modified. One has

$$x_0 = \sigma(\Theta) \Leftrightarrow (\forall t_1) x_0 t_1 = t_1$$

in the theory of monoids, defining the identity operation σ over the multiplication Σ, but Σ-homomorphisms do not preserve σ.

For the proofs, let T be a varietal theory and Σ a class of its operations. (A varietal theory [3] is allowed to have a proper class of operations; the crucial requirement is that a free algebra on any set of generators exists, as a set.) A Σ-*homomorphism* of T-algebras means a mapping preserving all the operations of Σ.

THEOREM 1. *If the m-ary operation σ is preserved by Σ-homomorphisms of T-algebras then there exists a set Φ of equations in the operations of Σ, in variables, y, $x_\alpha(\alpha < m)$, and $t_\beta(\beta \in B)$ such that $y = \sigma(\{x_\alpha\})$ in a T-algebra if and only if for some values of $t_\beta(\beta \in B)$, Φ holds.*

PROOF. Form a free T-algebra M on m generators x_α; list all its (n) elements, the generators as indicated, the element $\sigma(\{x_\alpha\})$ being called y, and the other elements being called $t_\beta(\beta \in B)$. Let N be a free T-algebra on n generators. Let W be the set of all elements of N which can be generated from the free generators by means of operations in Σ. For each element $\bar{w}$ of W select a Σ-word w expressing $\bar{w}$ in terms of the generators. Now there is a T-homomorphism $h: N \to M$ taking the generators bijectively to M. Let Φ consist of the equations, one for each $\bar{w}$ in W, giving the value $h(\bar{w})$ of the word w on the arguments y, x_α, t_β.

For any elements y^* and $x_\alpha^*(\alpha < m)$ of a T-algebra A, the solutions of Φ with $x_\alpha = x_\alpha^*$ correspond precisely to the Σ-homomorphisms $M \to A$ taking the generators to the x_α^*. (For any Σ-operation, τ applied to arguments at most n of which are different yields an element of W and thus a word w accounted for in Φ. The agreement of this specialization of τ with w is a theorem of T since it occurs for the generators of the free algebra N.) If $y^* = \sigma(\{x_\alpha^*\})$, there is a solution of Φ with $y = y^*$, $x_\alpha = x_\alpha^*$, given by a T-homomorphism. By hypothesis, all Σ-homomorphisms of M agreeing on $\{x_\alpha\}$ agree at y; so if $y^* \neq \sigma(\{x_\alpha\})$, there is no such solution of Φ.

PROPOSITION 1. *In any algebra, if Φ is a system of equations in variables y, $x_\alpha(\alpha \in A)$, and $t_\beta(\beta \in B)$, such that for all values of $x_\alpha(\alpha \in A)$ there is a unique value $\sigma(\{x_\alpha\})$ of y such that for some values of $t_\beta(\beta \in B)$, Φ holds, then $\sigma(\{x_\alpha\})$ is* *also defined as that* u *such that for all values of* y *and* t_β $(\beta \in B)$, $\Phi \Rightarrow y = u$.

Conversely, if the second formula defines a single-valued function, so does the first.

PROPOSITION 2. *The converse of Theorem* 1 *holds.*

Proposition 1 is trivial. Perhaps a bit deeper than its proof is the remark that it is not quite true for arbitrary formulas Φ. In a one-element algebra, $t \neq t \Rightarrow y = u$ defines a single-valued function u, but the corresponding existential formula does not. However, Proposition 1 is easily checked in the three cases that the number of elements is 1, or greater, or less.

For Proposition 2, a Σ-homomorphism takes solutions of Φ to solutions of Φ.

A theory is said to have *rank* r if r is the least regular cardinal $\geqq 2$ such that every element of any free T-algebra is in a subalgebra generated by fewer than r of the free generators [3]. Let us define the *raised rank* (if the rank exists) to be $r^* = \max(r, \aleph_0)$. It is known, and easily checked, that r^*-directed direct limits of T-algebras are T-algebras in the natural way. Hence:

THEOREM 2. *If the theory* T *has rank* r *then the set* Φ *of Theorem* 1 *may be taken to consist of fewer than* r^* *equations, which necessarily actually involve fewer than* r^* *variables.*

PROOF. Consider the r^*-directed set of subsets F of Φ of power less than r^*. When $y = \sigma(\{x_\alpha\})$, Φ and every F have solutions. Thus, were Theorem 2 false, there would be algebras A_F containing families $\{x_\alpha^F\}$ and elements $y^F \neq \sigma(\{x_\alpha^F\})$ such that for some t_β^F, F holds. In an obvious reduced product we get a counterexample to Theorem 1, which is absurd.

It seems worth noting that if just Σ and σ are finitary, not much follows. Consider the theory T of m-complete Boolean algebras with a complete Boolean algebra U of at most m unary operations $a(\ )$ subject to $[Va_i](x) = Va_i(x)$, $[\Lambda a_i](x) = \Lambda a_i(x)$. If $a = Va_i$ and the complement a' is Vb_j, then $a(x)$ is the unique y such that $y\Lambda 1(x) = y$, $y\Lambda a_i(x) = a_i(x)$, $y\Lambda b_j(x) = 0(x)$; $a(\ )$ is definable over Λ and a dense subalgebra of U, but not finitely. True, we have here no more equations than the number of elements of a free T-algebra on one generator. A weaker estimate always applies, as is clear from the proof of Theorem 1.

Finally we note that for theories of rank 2, the systems Φ are known in a more special form, and it is known that definable operations σ (always preserved, in this case) cannot be defined with a bounded number of variable symbols [1].

REFERENCES

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